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# Wave trapping above a plane beach by partially or totally submerged obstacles

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Trapped waves generated by oscillatory sources or dipoles placed above a plane infinite beach are examined within the framework of a (classical) non-hydrostatic but linear theory. This is achieved by solving a boundary-value problem where the boundary conditions are specified on the free surface and on the bottom. Integral expressions are derived for the complex potential for the cases where the sources or dipoles are strategically positioned to mimic the presence of solid bodies, a phenomenon manifested by the observation of a streamline enclosing the source or dipole. The precise positioning is governed by the further requirement of no radiating waves and, for the case where the beach is a vertical cliff, some recent results are confirmed here, whilst new results obtained show that infinitely many submerged wave trapping bodies exist and do so over a far greater range of values of dipole positions than was previously thought to be the case. The situation for surface sources and for submerged dipoles is therefore essentially different. For the former, infinitely many closed streamlines exist for each of the denumerably infinite set of source positions. For the latter, it is found instead that only one closed streamline exists, but this is for each of a non-denumerably infinite set of dipole positions. The expressions obtained for the beach are used for the two cases of a surface source and a submerged dipole to compute streamlines and stagnation points for model beaches of chosen steep slope. In particular, a (randomly chosen) submerged closed streamline is calculated for the beach of angle 45° thereby establishing a new case of non-uniqueness for the water wave problem on a beach.

## 1. Introduction

Wave trapping by obstacles has recently attracted the attention of many authors and an overview of recent results is given in McIver (1996) and McIver (2000) (see also Evans & Kuznetsov 1997). Hitherto, the work mostly appears to have treated problems for infinite or uniform depth and little recent attention appears to have been given to the case of a sloping bed. This (linear) problem was, however, originally discussed by Morris (1974*a*) for the case where an oscillating logarithmic source is placed at a zero of the regular potential $\phi_1$ (see Stoker 1957, p. 77–84) above a plane infinite beach. Following that work, Morris (1974*b*) proposed that, in such a case, where the wave field at infinity will be asymptotically small, closed streamlines can be found to enclose various positions of the source. The implication is then that there can exist, in principle, a trapped wave between the shore and an obstacle whose surface coincides with any of these stream lines. Thus this provides examples of non-uniqueness for the equivalent water wave problem although Morris did not compute any such closed streamlines.

Some difficulties were encountered (by this author) with the asymptotics given in Morris' papers, making numerical computations extremely unstable, and the intention is here to examine a different solution technique with the particular objective of giving more robust asymptotics. It is shown that the Mellin transform approach provides this technique.

The formulation is begun with the simple case of a 45° beach where the source is placed on the surface. This is then generalized to beach angles of the form $\pi/2k$, $k \geqslant 1$ and (for ease of computation) evaluated for some special cases where $k$ is integer. These computations include the calculation of streamlines surrounding the source positions, chosen to eliminate radiating waves.

The solution is then further generalized to the case where the source is submerged thus enabling immersed oscillating dipole solutions also to be constructed. The conditions for zero radiation are maintained for all cases and numerical computation undertaken for a range of parameters. In so doing, the numerical results of McIver (2000) are recovered for her case ($k = 1$) which is here equivalent to the beach becoming a vertical cliff. In that work, she demonstrated the existence of a submerged trapping obstacle.

One of the main aims of the present work is the generalization of McIver's work to beaches of arbitrary slope. This is carried out by demonstrating numerically, on a beach of unit gradient, the existence of the essential ingredient, namely the occurrence of two submerged stagnation points having a common stream function value. Indeed, it turns out that, both for the cliff and the beach of gradient unity, this situation can be constructed for dipole positions lying on a (submerged) trajectory whose dimensionless horizontal coordinate seems to be restricted only to sets of intervals of lengths $\pi/2$. There seems no reason to suppose that there is a restriction on the number of these intervals, but here only the existence of the first two are demonstrated in the two cases considered. Having established the above, it is then a matter only of routine computation to calculate the unique submerged closed streamline which surrounds the chosen dipole position. This is done in a number of cases.

The layout of the paper is as follows. In the next section the procedure for the simplest possible case, namely a source on the surface with a beach of gradient unity, is outlined. The fundamental technique of using the inverse Mellin transform is developed whereby the surface Robin condition results in a first-order difference equation for the Mellin kernel. The method of solving this is outlined in § 3 and it is verified, in § 4, that non-radiating solutions are provided by the condition that the source is placed at a zero of potential for the classical scattering problem without a source. This fact can easily be verified by an application of Green's theorem, but the verification through the solution itself provides an excellent check on the ansatz and the way the solution is developed. Some observations on numerical inversion of the Mellin transform and the asymptotics of the integrands are described in § 5 and this is followed in §§ 6, 7 and 8 respectively by the generalization to arbitrary beach angles, descriptions for submerged sources and finally submerged dipoles. This last section includes a computation for the case of a vertical cliff, which establishes the absolute agreement with previous results by McIver (2000). The application of the theory to a beach is explored in § 9 with a number of examples for a beach of unity gradient. The observation is made there that other beach angles may be similarly dealt with, although obviously the computations will be somewhat more demanding in terms of time.

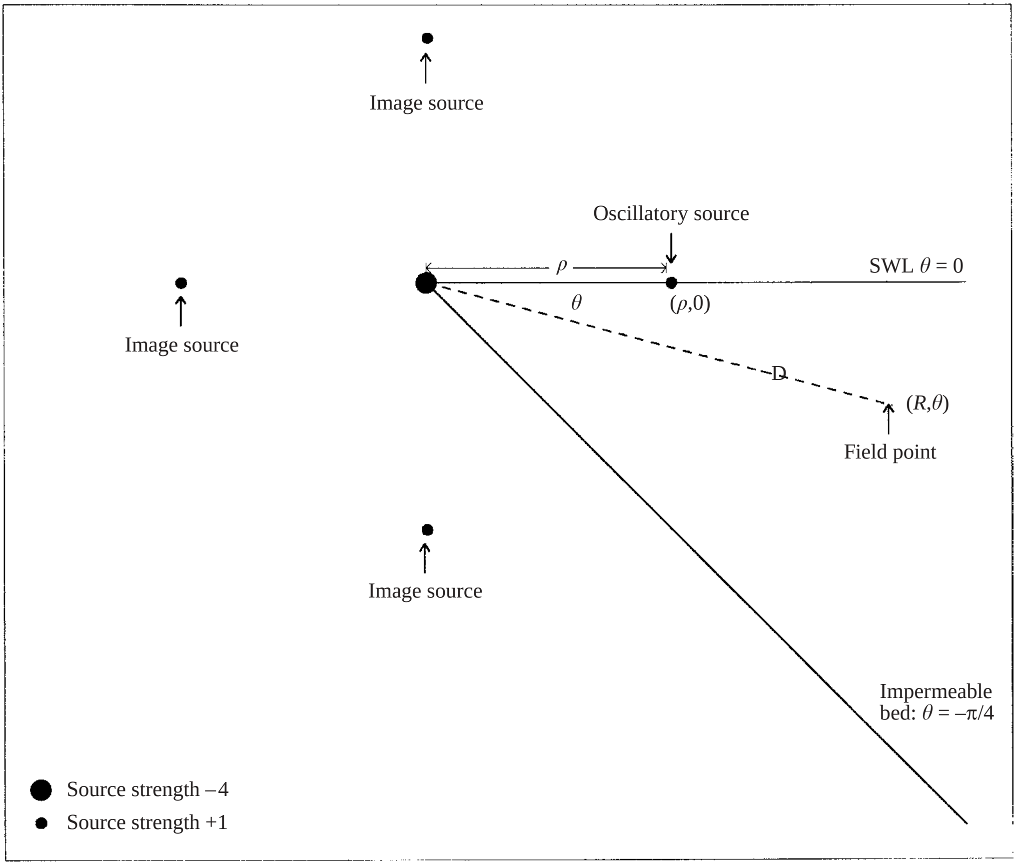


FIGURE 1. Non-dimensionalized coordinates and image system. Case: surface source at distance $\rho$ from shoreline; $\alpha = \pi/4$.

## 2. Ansatz

Polar coordinates $r, \theta$ are useful for this type of problem with the polar line $r = 0$ representing the shoreline, $\theta = 0$ the still-water level and $\theta = -\alpha$ the rigid bottom, so that the wedge so formed represents the primary domain D of the flow (figure 1). Periodic wave motion at angular frequency $\omega$ will be generated by sources and dipoles and this enables the entire problem to be non-dimensionalized with respect to the length $g/\omega^2$ and the time $\omega^{-1}$. Throughout this work it is assumed that the non-dimensional velocity potential $\Phi$ and stream function $\Psi$ are given by

$$(\Phi, \Psi) = \mathrm{Re}((\phi(R, \theta), \psi(R, \theta))\mathrm{e}^{\mathrm{i}\omega t}),$$

where $R = (\omega^2/g)r$. The governing equations (see also Morris 1974*a*) are first written (and subsequently generalized) for the case of a 45° beach, here with the single source positioned on the surface $\theta = 0$ at a distance $R = \rho$ from the shoreline $R = 0$ but later in the work with either source or dipole at submerged positions. The equations are

$$\nabla^2\phi = 0, \quad (R, \theta) \in \mathrm{D}, \tag{2.1}$$

$$\frac{\partial\phi}{\partial\theta} = 0, \quad \theta = -\alpha, \tag{2.2}$$

$$\frac{1}{R}\frac{\partial\phi}{\partial\theta} = \phi, \quad \theta = 0, \tag{2.3}$$

$$\phi \text{ bounded as } R \to 0, \tag{2.4}$$

(2.5)

and, for the correct singularity at the source,

$$\phi \to \log|z-\rho|, z \to \rho, \tag{2.6}$$

where $z = Re^{i\theta}$. Instead of using the image system defined by Morris (1974*a*), a quadrupole-type arrangement is taken with sources +1 at points $z = \pm 1, \pm i$ together with a source −4 at the origin; here $z = Re^{i\theta}$. Whilst this seems to introduce a logarithmic singularity at the origin (which will need to be removed) it will yield a much simpler Mellin transform analysis than would Morris' arrangement.

A solution to the problem is then expressed in the form

$$\phi(R,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s}A(s)\cos s(\theta+\alpha)\,ds + \log\left|\frac{z^4-\rho^4}{z^4}\right|, \tag{2.7}$$

where $c$ is a suitable real constant. The condition on the bed is satisfied automatically and the free-surface condition $L_1[\phi]|_{\theta=0} = 0$, where $L_1 \equiv R - \partial_\theta$, can be expressed in the form

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s}A(s)\{\cos s\alpha + (s/R)\sin s\alpha\}\,ds + \log\left|\frac{R^4-\rho^4}{R^4}\right| = 0$$

subject to differentiability under the integral sign. It is assumed that $0 < c < 1$ and consequently required that $A(s)$ has precisely a double pole at the origin in order to balance the logarithmic singularity at $R = 0$. Subject to suitable behaviour at infinity the second of the two integrals can therefore, by Cauchy's theorem, be written $(1/2\pi i)\int_{c-1-i\infty}^{c-1+i\infty} R^{-\tau-1}\tau A(\tau)\sin(\tau\alpha)\,d\tau$ provided $A(s)$ is otherwise analytic in the strip $c-1 \leqslant \mathrm{Re}\,s \leqslant c$. Following the substitution $\tau = s-1$, the two integrals can now be combined as one single inverse Mellin transform. Rectification of the inverse then results in the difference equation

$$A(s)\cos s\alpha + (s-1)A(s-1)\sin(s-1)\alpha = -\frac{\rho^s}{4}\int_0^\infty \tau^{s/4-1}\log\left|\frac{\tau-1}{\tau}\right|d\tau, \tag{2.8}$$

and the assumptions made are verified *a posteriori* in connection with boundedness requirements of the physical solution $\phi$.

Using results (7.7.10) and (7.7.17) given by Titchmarsh (1948) the integral on the right-hand-side is found to be $(4\pi/s)\cot(\pi s/4)$.

## 3. Solution

The difference equation may be solved by a standard method following a simple transformation. Thus, with $\Gamma(s)$ denoting the usual gamma function, put

$$A(s) = \Lambda(s)\Gamma(s)\sin(s-1)\alpha$$

so that the equation transforms to

$$\Lambda(s) - \Lambda(s-1) = \cot\left(\frac{\pi s}{4}\right)\frac{\pi\rho^s}{\Gamma(s+1)\sin(s-1)\alpha\sin(s-2)\alpha}.$$

Note that $\Lambda(s)$ is now required to have a simple pole at $s = 0$. Accordingly write

$$\Lambda(s) = \frac{\lambda}{1 - e^{2\pi i s}} + C(s)$$

where $\lambda$ is suitably chosen to regularize the potential at $R = 0$ and $C(s)$ satisfies the same difference equation as $\Lambda$. A particular solution can then be constructed from the contour integral

$$C(s) = \int_L \frac{C(\tau)e^{2\pi i\tau}}{e^{2\pi i\tau} - e^{2\pi i s}}\, d\tau$$

provided $L$ is suitably chosen and contains only the one simple pole at $\tau = s$. Assuming that, in the outer solution integral, $\mathrm{Re}(s) = \frac{1}{2}$ (see e.g. Ehrenmark 1987) such a choice for $L$ is given by the boundary of a rectangle whose sides are $\mathrm{Re}(\tau) = -\frac{1}{4}, \frac{3}{4}$ and $\mathrm{Im}(\tau) = \pm Y$ where $Y$ will be taken arbitrarily large. The resulting solution may then be written

$$C(s) = \int_{3/4-i\infty}^{3/4+i\infty} \cot\left(\frac{\pi\tau}{4}\right) \frac{\pi\rho^\tau e^{2\pi i\tau}}{\Gamma(\tau+1)\sin(\tau-1)\alpha\sin(\tau-2)\alpha} \left\{\frac{1}{e^{2\pi i\tau} - e^{2\pi i s}}\right\} d\tau.$$

Note, in particular that $C(s)$ is regular at $s = 0$ and that, by shifting the integration contour between $\mathrm{Re}\tau = \delta$ and $\mathrm{Re}\tau = 1 - \delta$, $0 < \delta < 1$, it can be argued that $C(s)$ is indeed analytic in $-1 + \delta < \mathrm{Re}(s) < 1 - \delta$ so that, by Montel's theorem (Titchmarsh 1939, p. 170), any asymptotic limit deduced for $C(s)$, $|\mathrm{Im}(s)| \to \infty$ will be valid in the entire strip. The asymptotics of $C(s)$ which justifies the vanishing of the contributions to the contour integral over the paths $\mathrm{Im}(\tau) = \pm Y$ is discussed in the next section. It is required in particular that $|C(s)|$ remains bounded as $\mathrm{Im}s \to +\infty$.

## 4. Non-radiating solution

The near- and far-field asymptotics of inverse Mellin transforms of the type in equation (2.7) has been discussed by the author in several papers, e.g. Ehrenmark (1987) where it is demonstrated that a radiating wave field will result in an inversion integral with the property of convergence at infinity as a Cauchy principal value. In the absence of a radiating field therefore, it is expected that the integral will be at least conditionally convergent and that, consequently, $|A(s)\cos s\alpha|$ will be $O(|y|^{-1+\nu}|)$, $\nu \geqslant 0$ as $y \to \pm\infty$ on $s = \frac{1}{2} + iy$.

Clearly, as $\mathrm{Im}s \to -\infty$ there is no difficulty with $A(s)$ exponentially small. When $\mathrm{Im}s \to +\infty$, the result, after passing the integral for $C(s)$ over the simple pole at $\tau = 0$, is

$$\Lambda(s) \sim \lambda + 8\pi i\sqrt{2} + f(\rho) + O(|s|^{-1})$$

where

$$f(\rho) = \int_{-1/2-i\infty}^{-1/2+i\infty} \cot\left(\frac{\pi\tau}{4}\right) \frac{\pi\rho^\tau}{\Gamma(\tau+1)\sin(\tau-1)\alpha\sin(\tau-2)\alpha}\, d\tau,$$

and the error term is verified in the next section. This last expression can be transformed into the (classical scattering potential) solution integral for the regular wave in the absence of a source (Ehrenmark 1987) by making the substitution $\tau = -t$, noting that $\alpha = \pi/4$ and that $\Gamma(t)\sin\pi t = \pi/\Gamma(1-t)$. This expression is

$$f(\rho) = \int_{1/2-i\infty}^{1/2+i\infty} \rho^{-t}\Gamma(t)\cos t\alpha\sin(t-1)\alpha\, dt.$$

So for a non-radiating field, $\rho$ needs to be chosen so that

$$f(\rho) = -(\lambda + 8\pi \mathrm{i}\sqrt{2}).$$

(One can also reason this from arguments involving the Riemann–Lebesgue lemma.) The value of $\lambda$ is governed by the requirement of boundedness at the shoreline. The near-field asymptotics of the solution of (2.7) is determined by the residues of the integral in the left-hand half-plane and in particular, therefore, the double pole at the origin which yields the logarithmic term. Putting $s = \epsilon$, for small $\epsilon$, the integrand expands like

$$\frac{(\Gamma(1) + \epsilon\Gamma'(1))(1 - \epsilon \log R)(-\sin\alpha + \epsilon\alpha\cos\alpha)\lambda}{-2\pi \mathrm{i}\epsilon^2(1 + O(\epsilon))},$$

giving the contribution to the logarithmic ($\log R$) part arising from the residue $-\lambda \sin\alpha \log R/(2\pi \mathrm{i})$. If the logarithmic term from the quadropole arrangement is added, it is found that the choice $\lambda + 8\pi\mathrm{i}\sqrt{2} = 0$ leads to removal of logarithmic terms at the shoreline. Moreover, this also leads to $f(\rho) = 0$ so that the chosen positions of the surface source must coincide with zeros of the regular scattering potential, in precise agreement with the results of Morris (1974*a*, *b*). This result of course can also be obtained by applying Green's identity in the usual way to the two potentials $\phi$ and $\phi_0$, where $\phi_0$ is the classical scattering potential and $\phi$ that of the present problem. It therefore provides an excellent check on the validity of the development of the solution for $C(s)$ and the ansatz used.

## 5. Numerical procedure

The stream function for this non-radiating field is given by

$$\psi(R, \theta) = -\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} R^{-s} A(s) \sin s(\theta + \alpha)\, \mathrm{d}s + \arg\left\{\frac{z^4 - \rho^4}{z^4}\right\}. \tag{5.1}$$

The first numerical expedient is to note that (entirely as expected) the integrand is purely real on the real axis (this may be observed, for example, by considering the residue calculation of the integral for $C(s)$ completing a contour in the left-hand half-plane). The expression for $C(s) + \lambda/(1 - \mathrm{e}^{2\pi\mathrm{i}s})$ can be rewritten as

$$\Lambda(s) = 8 \int_{1/4-\mathrm{i}\infty}^{1/4+\mathrm{i}\infty} \frac{\rho^{-\tau}\Gamma(\tau)\cos\tau\alpha \sin(\tau - 1)\alpha}{1 - \mathrm{e}^{2\pi\mathrm{i}(s+\tau)}}\, \mathrm{d}\tau - 8\Gamma(-s)\rho^s \cos s\alpha \sin(s + 1)\alpha.$$

This can be evaluated by residues in the left-hand half-plane, providing a solution which could have been obtained rather more trivially (see e.g. Titchmarsh 1948, p. 302). The set arising from the poles $\tau = -s - N, N = 0, 1, \ldots$ gives the final expression

$$\Lambda(s) = 8 \sum_{N=1}^{\infty} \rho^{s+N}\Gamma(-s - N) \cos(s + N)\alpha \sin(s + N + 1)\alpha, \qquad \mathrm{Re}\, s > -\tfrac{1}{2} \tag{5.2}$$

because the set given by the poles of $\Gamma(\tau)$ contributes a fixed multiple $8/(1 - \mathrm{e}^{2\pi\mathrm{i}s})$ of the vanishing integral $f(\rho)$. Thus $\Lambda(s)$ is real on the real axis and the simplification suggested is substantiated. Note also from equation (5.2) that the sum provides a Poincaré asymptotic expansion of $\Lambda(\frac{1}{2} + \mathrm{i}t)$, $t \to +\infty$, the first term of which is $O(t^{-2})$. Accordingly, equation (5.1) may be written

$$\psi(R,\theta) = \frac{-1}{\pi\sqrt{R}} \mathrm{Re} \int_0^\infty R^{-\mathrm{i}t} \Gamma\left(\tfrac{1}{2}+\mathrm{i}t\right) \Lambda\left(\tfrac{1}{2}+\mathrm{i}t\right) \sin\left(-\tfrac{1}{2}+\mathrm{i}t\right)\alpha$$
$$\times \sin\left(\tfrac{1}{2}+\mathrm{i}t\right)(\theta+\alpha)\,\mathrm{d}t + \arg\left\{\frac{z^4-\rho^4}{z^4}\right\}. \quad (5.3)$$

From the point of view of numerical integration it is useful to combine where possible terms that (as $t \to \infty$) grow large with those that decay. One convenient way of doing this here is to exploit the identity

$$\Gamma\left(\tfrac{1}{2}+\mathrm{i}\tau\right)\Gamma\left(\tfrac{1}{2}-\mathrm{i}\tau\right) \equiv \pi \operatorname{sech} \pi\tau$$

and factor $\Gamma(1-s)$ from the summation expression for $\Lambda(s)$. Use also of the identity $\Gamma(-s+1) = s(s+1)\Gamma(-s-1)$ shows that on the free surface each of the integrals for $N = 1, 2, \ldots$ is absolutely convergent with integrands of order $t^{-1-N}$ whilst in the fluid interior there is superposed exponential decay also. Numerical integration is therefore comparatively straightforward.

## 6. General beach angles

In the case of general values of $\alpha = \pi/2k < \pi/2$ the Schwarz–Christoffel transformation on the image system potential provides an equivalent description (cf. equation (2.7)) in the form

$$\phi(R,\theta) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} R^{-s} \tilde{B}(s)\Gamma(s)\cos s(\theta+\alpha)\,\mathrm{d}s + \log\left|\frac{z^{\pi/\alpha}-\rho^{\pi/\alpha}}{z^{\pi/\alpha}}\right|. \quad (6.1)$$

The difference equation for $\tilde{B}$ is established essentially as before and is

$$\tilde{B}(s)\cos s\alpha + \tilde{B}(s-1)\sin(s-1)\alpha = -\pi\rho^s \frac{\cot s\alpha}{\Gamma(s+1)}. \quad (6.2)$$

In order to reduce the left-hand side to the simple difference operator there is a need to be reminded of the fundamental Mellin transform solution to the classical wave problem (Ehrenmark 1987). The bounded solution with unit amplitude at infinity is given by

$$\phi_0(R,\theta) = \frac{1}{2\pi \mathrm{i}} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} R^{-s} B(s)\Gamma(s)\sin\pi s \frac{\cos s(\theta+\alpha)}{\sqrt{2\pi}\cos s\alpha}\,\mathrm{d}s \quad (6.3)$$

where $B(s)$ is defined by

$$B(s) = \Gamma(s)\exp\left[\int_0^\infty \frac{\mathrm{d}t}{t}\left\{\frac{2\mathrm{e}^{t/2}\sinh\left(s-\tfrac{1}{2}\right)t}{(\mathrm{e}^{kt}+1)(\mathrm{e}^t-1)} - \left(s-\tfrac{1}{2}\right)\mathrm{e}^{-t}\right\}\right], \quad -k < \mathrm{Re}\, s < k+1. \quad (6.4)$$

Note also that $B(s+1) = B(s)\tan s\alpha$. A closed form is obtained when $k$ is integer:

$$B(s) = 2^{k-1}\sqrt{2\pi}\operatorname{cosec}\pi s \prod_{j=0}^{k-1}\cos(s+j)\alpha, \quad 0 < \mathrm{Re}\, s < 1 \quad (6.5)$$

and it may be noted that (on $\theta = 0$) a multiple of $f(\rho)$ is recovered in the case $k = 2$. An important consequence of equation (6.4) is that

$$B(s)B(1-s) = \pi\operatorname{cosec}\pi s.$$

The substitution

$$\tilde{B}(s)\cos s\alpha = B(s)\sin \pi s\, d(s)$$

therefore reduces the difference equation (6.2) to

$$d(s) - d(s-1) = -\frac{\pi\rho^s \cot s\alpha}{\Gamma(s+1)B(s)\sin\pi s}. \tag{6.6}$$

This can now be solved by Titchmarsh's procedure which is tantamount to forming the formal sum and then confirming convergence. This will be assured in $\mathrm{Re}\,s > 0$ for the case where $k$ is integer (see the Appendix for a fuller explanation). The simple pole of $\tilde{B}$ required at $s = 0$ (see § 2) again requires that $sd(s)$ must be regular at the origin. It can be concluded that a full solution to the difference equation may therefore be given by

$$d(s) = \frac{\tilde{\lambda}}{1 - \mathrm{e}^{2\pi i s}} + \sum_{N=1}^{\infty} \frac{\pi\rho^{(s+N)}\cot(s+N)\alpha}{\Gamma(s+N+1)B(s+N)\sin\pi(s+N)}, \qquad 0 < \mathrm{Re}(s) < 1, \tag{6.7}$$

where $\tilde{\lambda}, \rho$ are chosen as before to ensure regularity at the shoreline and non-radiating field at infinity. This form, although useful computationally, is not an optimum for discussing singularities. The equivalent integral solution is given by $d(s) = \lambda/(1 - \mathrm{e}^{2\pi \mathrm{i} s}) + \tilde{d}(s)$ where the particular integral is

$$\tilde{d}(s) = -\int_{3/4-\mathrm{i}\infty}^{3/4+\mathrm{i}\infty} \frac{\pi\cot\tau\alpha\rho^{\tau}\mathrm{e}^{2\pi\mathrm{i}\tau}}{\Gamma(\tau+1)B(\tau)\sin\tau\pi}\left\{\frac{1}{\mathrm{e}^{2\pi\mathrm{i}\tau} - \mathrm{e}^{2\pi\mathrm{i}s}}\right\}\mathrm{d}\tau.$$

Proceeding as before (passing over poles at $\tau = 0$ and $\tau = s$) the expression can be rewritten as

$$\tilde{d}(s) = \int_{1/4-\mathrm{i}\infty}^{1/4+\mathrm{i}\infty} \frac{\rho^{-\tau}B(\tau)}{\Gamma(1-\tau)}\left\{\frac{1}{1 - \mathrm{e}^{2\pi\mathrm{i}(s+\tau)}}\right\}\mathrm{d}\tau - \frac{2\pi\mathrm{i}}{\alpha\epsilon B(\epsilon)(1 - \mathrm{e}^{2\pi\mathrm{i}s})} - \frac{\pi\cot s\alpha\,\rho^s}{\Gamma(1+s)B(s)\sin\pi s}$$

where the limit $\epsilon \to 0$ is understood. Denote

$$\pi f_k(\rho) = \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \rho^{-\tau}B(\tau)\Gamma(\tau)\sin\pi\tau\,\mathrm{d}\tau,$$

then, as $\mathrm{Im}\,s \to +\infty$, it follows that

$$d(s) \sim \lambda - \frac{2\pi\mathrm{i}}{B(1)} + f_k(\rho)$$

so $\lambda$ has to be chosen to make this vanish in order to have a non-radiative wave field. The asymptotics required to ensure that the last term in the expression for $\tilde{d}$ is $O(|s|^{-1})$ is provided by the result $|B(\frac{1}{2} + \mathrm{i}\tau)|^2 = \pi/(\cosh\pi\tau)$ and the asymptotics of $\Gamma(1+s)$ on $s = \frac{1}{2} + \mathrm{i}\tau$.

Meanwhile, considering again the residue from the double pole at $s = 0$ in the solution integral (equation (6.1)) one finds a term $(\lambda B(1)/(2\mathrm{i}\alpha) - \pi/\alpha)\log R$. The value of $B(1)$ is found to be $\sqrt{\alpha}$, so the choice of $\rho$ governed by

$$\lambda = \frac{2\mathrm{i}\pi}{\sqrt{\alpha}}$$

leads again to the source positions of the zeros of the regular potential solution, i.e. $f_k(\rho) = 0$. The outcome of this is equivalent to choosing $\tilde{\lambda} = 0$ so that a

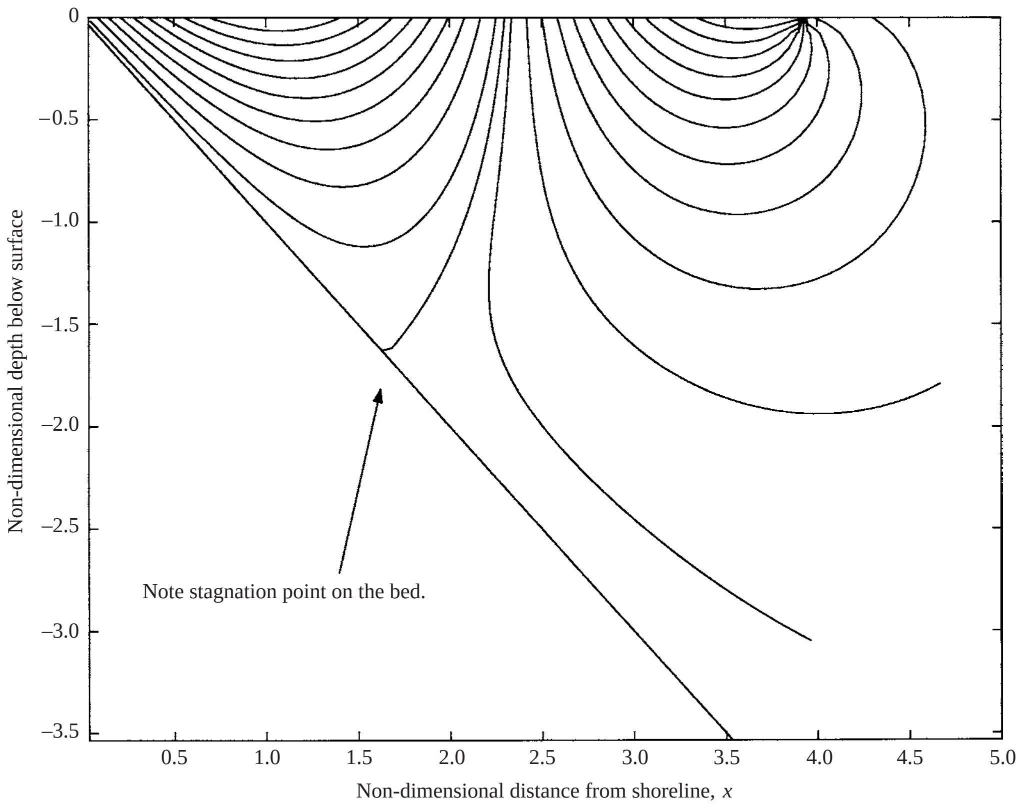


FIGURE 2. Streamlines around surface source. Case: $\alpha = \pi/4$, with oscillating source on the surface at $x = 3.9$.

computationally robust strategy (on $s = \frac{1}{2} + \mathrm{i}y$) can be founded on the solution

$$\tilde{B}(s) = -\frac{B(s)\sin \pi s}{\cos s\alpha} \sum_{N=1}^{\infty} \frac{\rho^{s+N}\Gamma(-s-N)}{B(1+s+N)},$$

for the case where $k$ is integer. In more general cases, the zeros and singularities of $B(s)$ are somewhat complicated and a summation strategy is not prudent. In such cases the integral expression for $d(s)$ should be used. Reverting to integer $k$, the full stream function is therefore expressible as

$$\psi = \frac{-1}{2\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left(\frac{R}{\rho}\right)^{-s} \frac{\sin s(\theta+\alpha)}{\cos s\alpha} \sum_{N=1}^{\infty} (-\rho)^N \prod_{j=0}^{N} \frac{\cot(s+j)\alpha}{(s+j)} \, \mathrm{d}s + \arg\left\{\frac{z^{\pi/\alpha} - \rho^{\pi/\alpha}}{z^{\pi/\alpha}}\right\}. \tag{6.8}$$

## 6.1. *Computations for a source on the surface*

Calculations are presented by means of streamline diagrams for various source positions. In each of these cases the source has been placed at a zero of the classical potential solution, consistent with the requirement for zero radiation. In each case it may also be noted that there are closed streamlines surrounding the source, thus confirming the existence of wave-trapping solid objects with the surface piercing property. Note further the existence near the bed of stagnation points. The cases $\alpha = \pi/4$ and $\alpha = \pi/8$ have been chosen for simplicity. Figures 2 and 3 show, in each

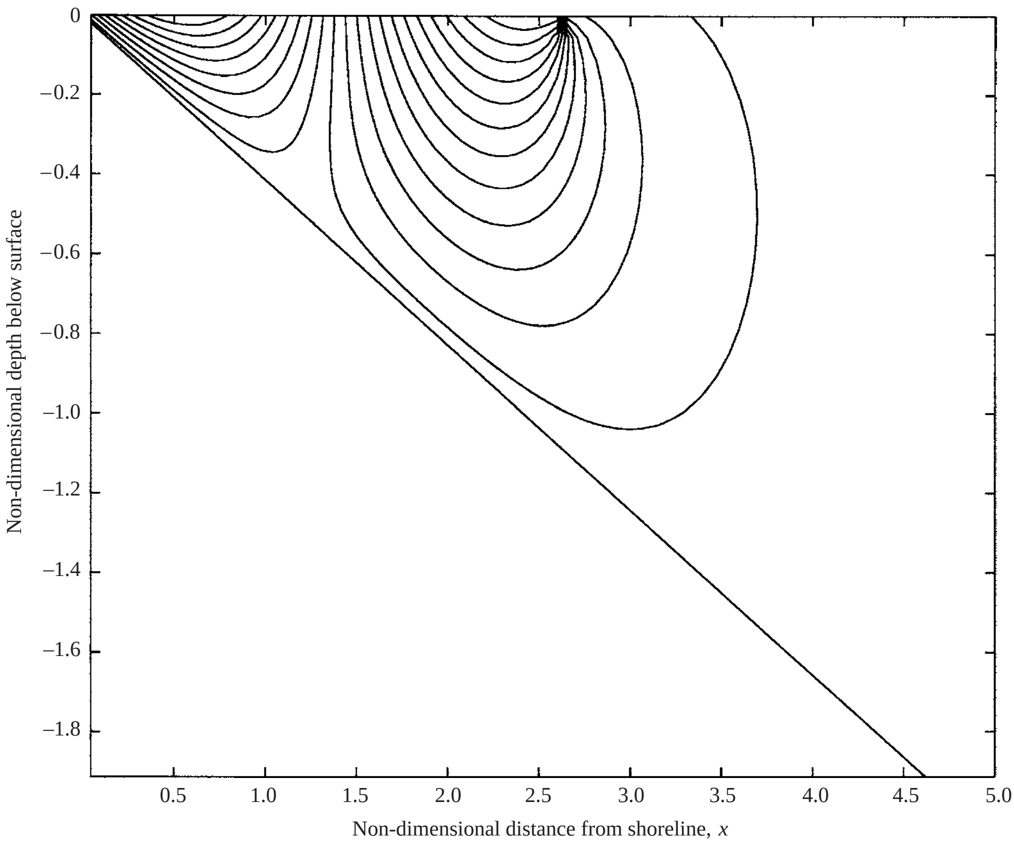


FIGURE 3. Streamlines around Mode 2 surface source. Case: $\alpha = \pi/8$, with oscillating source on the surface at $x = 2.6$.

case, the streamlines for the non-radiating case (i.e. the source is placed at a zero of $\phi_0$) for the second mode (mode is used here in the sense that mode $i$ means the $i$th zero of $\phi_0$ counting from the shoreline).

To show a higher mode, calculations have also been done for mode 3 in the case of the $\alpha = \pi/8$ beach. The streamlines for this case are displayed in figure 4. Finally, it seemed of interest to display the effective standing wave elevation profile when a chosen stream-line surrounding the surface source is replaced by a solid object. This is done in figures 5 and 6 for the respective modes 2 and 3 on the respective beach angles $\pi/4$, $\pi/8$.

## 7. A submerged source

In order similarly to consider the submerged source position $(\rho, -\gamma), \gamma > 0$, in polar coordinates, it is prudent to introduce first the image system shown in figure 7 for the case of beach angle $\alpha = \pi/4$. The following expression may then be used as an ansatz for the potential:

$$\phi(R, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R^{-s} \tilde{B}(s) \Gamma(s) \cos s(\theta + \alpha) \, ds + \log \left| \frac{(z^4 - \rho^4 e^{4i\gamma})(z^4 - \rho^4 e^{-4i\gamma})}{z^8} \right| .$$

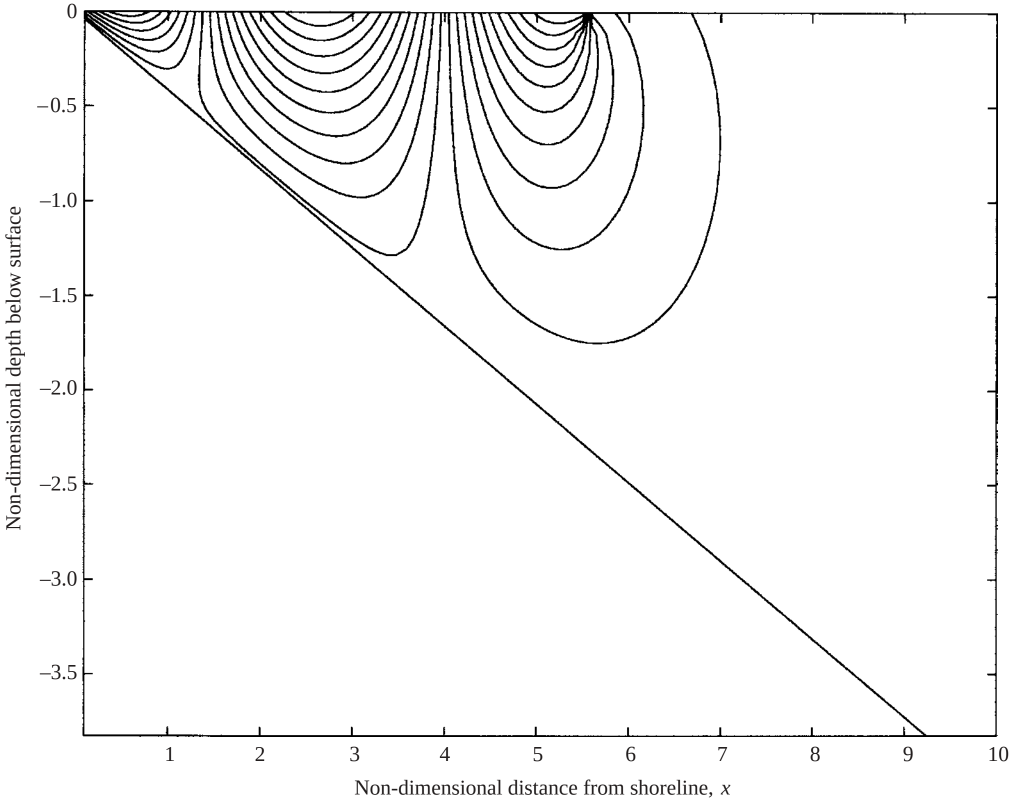


FIGURE 4. Streamlines around Mode 3 surface source. Case: $\alpha = \pi/8$, with oscillating source on the surface at $x = 5.52$.

This is easily generalized to any slope angle,

$$\phi(R,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s}\tilde{B}(s)\Gamma(s)\cos s(\theta+\alpha)\,ds + \log\left|\frac{\left(z^{\pi/\alpha} - \rho^{\pi/\alpha}e^{i\gamma\pi/\alpha}\right)\left(z^{\pi/\alpha} - \rho^{\pi/\alpha}e^{-i\gamma\pi/\alpha}\right)}{z^{2\pi/\alpha}}\right|. \quad (7.1)$$

The Mellin transform of the logarithmic term on $\theta = 0$ is required in order that the earlier analysis can be repeated. By denoting the first term on the right-hand side of equation (7.1) by $\phi^{(0)}$ and using an overbar to indicate the Mellin transform with respect to $R$, one can write

$$\overline{\phi}(s,0) = \overline{\phi^{(0)}}(s,0) + 2\int_0^\infty R^{s-1}\log\left|1 - \left(\frac{\rho}{R}\right)^{\pi/\alpha} e^{i\gamma\pi/\alpha}\right| dR.$$

To evaluate this Mellin transform, we again use results (7.7.10) and (7.7.17) given by Titchmarsh (1948) after splitting and transforming the integrand in the obvious way onto respectively $\arg R = \gamma$ and $\arg R = -\gamma$, following which integration can be restored onto the real axis (by contour deformation) in both cases. The result finally

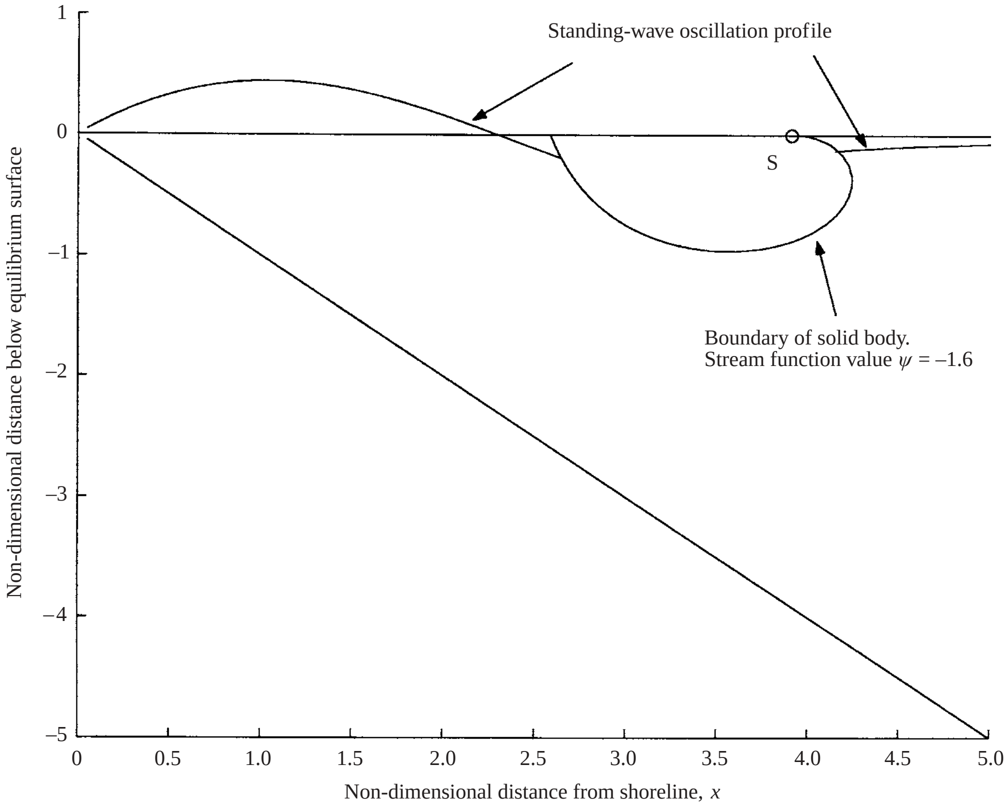


FIGURE 5. Elevation standing wave profile with solid body replacing a chosen streamline. Case: $\alpha = \pi/4$; Mode 2 non-radiating source S on surface at $x = 3.913$.

is that the difference equation (6.2) is modified to

$$\tilde{B}(s)\cos s\alpha + \tilde{B}(s-1)\sin(s-1)\alpha = -2\pi\rho^s \frac{\cos s(\gamma-\alpha)}{\sin\alpha s\ \Gamma(s+1)}, \tag{7.2}$$

whilst that for $d(s)$ is similarly modified to

$$d(s) - d(s-1) = -\frac{2\pi\rho^s\cos s(\gamma-\alpha)}{\sin\alpha s\ \Gamma(s+1)B(s)\sin\pi s}. \tag{7.3}$$

Its solution is then, as before, expressible in either of the alternative forms (for suitable $\lambda, \tilde{\lambda}$)

$$d(s) = \frac{\tilde{\lambda}}{1-\mathrm{e}^{2\pi i s}} + \sum_{N=1}^{\infty} \frac{2\pi\rho^{(s+N)}\cos(s+N)(\gamma-\alpha)}{\sin(s+N)\alpha\ \Gamma(s+N+1)B(s+N)\sin\pi(s+N)}, \quad 0 < \mathrm{Re}(s) < 1 \tag{7.4}$$

if $k$ is integer or

$$d(s) = \frac{\lambda}{1-\mathrm{e}^{2\pi i s}} - \int_{3/4-\mathrm{i}\infty}^{3/4+\mathrm{i}\infty} \frac{2\pi\cos\tau(\gamma-\alpha)\rho^\tau}{\sin\tau\alpha\ \Gamma(\tau+1)B(\tau)\sin\tau\pi}\left\{\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}(s-\tau)}}\right\}\mathrm{d}\tau,$$

for arbitrary $k$. The same procedure as before is carried out, passing over the poles at $\tau = 0$ and $\tau = s$ and noting the asymptotics of $d(s)$ as $\mathrm{Im}\, s \to +\infty$ remains

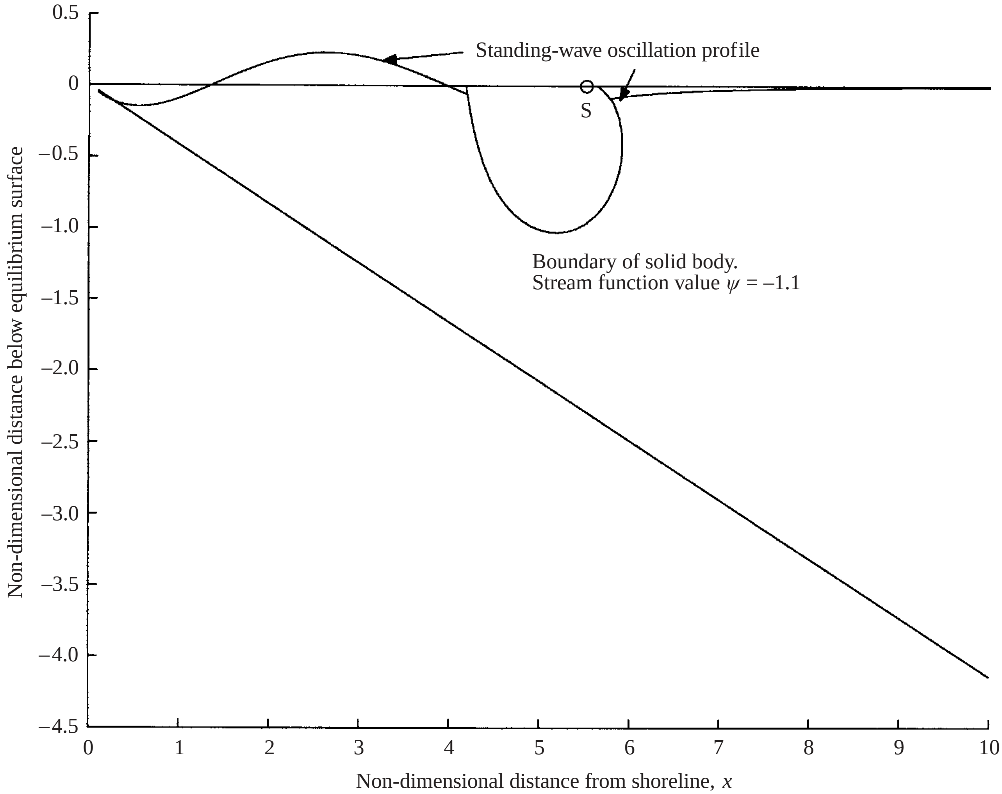


FIGURE 6. Elevation standing wave profile with solid body replacing a chosen streamline. Case: $\alpha = \pi/8$; Mode 3 non-radiating source S on surface at $x = 5.52$.

unchanged. The procedure for choosing the values of $\rho, \gamma$ so that $\phi_0(\rho, -\gamma) = 0$ (equation (6.3)) therefore again guarantees both the regularity of the solution (7.1) at $R = 0$ and the absence of radiating waves. (The residue from the double pole at $s = 0$ remains unchanged for the submerged source.) The expression for $\tilde{B}(s)$ is then modified to

$$\tilde{B}(s) = -\frac{2B(s)\sin \pi s}{\cos s\alpha} \sum_{N=1}^{\infty} \frac{\rho^{s+N}\Gamma(-s-N)\cos(s+N)(\gamma-\alpha)}{B(s+N)\sin(s+N)\alpha},$$

so that, for integer $k$, the expression for the stream function can be written

$$\psi = \mathrm{i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left(\frac{R}{\rho}\right)^{-s} \frac{\sin s(\theta+\alpha)}{s \sin s\alpha} \sum_{N=1}^{\infty} (-\rho)^N \frac{\cos(s+N)(\gamma-\alpha)}{\cos(s+N)\alpha} \prod_{j=1}^{N} \frac{\cot(s+j)\alpha}{(s+j)}\, \mathrm{d}s$$
$$+ \arg\left\{ \frac{\left(z^{\pi/\alpha} - \rho^{\pi/\alpha}\mathrm{e}^{\mathrm{i}\gamma\pi/\alpha}\right)\left(z^{\pi/\alpha} - \rho^{\pi/\alpha}\mathrm{e}^{-\mathrm{i}\gamma\pi/\alpha}\right)}{z^{2\pi/\alpha}} \right\}.$$

## 8. A submerged dipole

Green's theorem shows that, in a quest for a totally submerged closed streamline, it would be necessary to consider submerged sources in pairs of equal but opposite

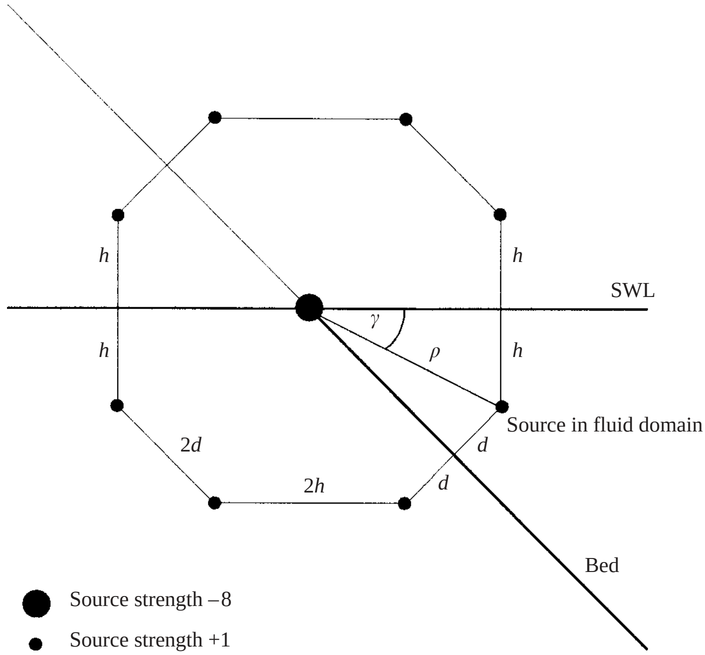


FIGURE 7. Typical image source distribution in the submerged case, $\alpha = \pi/4, d = \rho \sin(\alpha - \gamma)$.

strengths($\pm 1$). Some experiments have been carried out to this effect with such a pair, placing each on a different branch of the valid non-radiative trajectories (given by $\phi_0(\rho, -\gamma) = 0$). To date, these experiments have not yielded a positive result. However, in McIver (2000) it is shown that it is possible to insert an oscillating submerged dipole in front of a vertical barrier in such a way that a closed streamline surrounds the dipole and therefore, by implication, with no waves radiated, yet another non-uniqueness example is established. The case treated by McIver is that of infinite depth. The question therefore naturally arises of whether this can also be done for the case where the barrier is instead a plane beach. To this end, the generalization of equation (7.5) is required for the case where the submerged source is replaced by a dipole.

Clearly, with a submerged dipole, the image distribution has to take account of the angle of its axis. McIver chose equal-strength vertically and horizontally aligned dipoles, but with the inclined bed it is by no means clear that such an arrangement would be sufficient. Take therefore the dipole axis at angle $\beta$ to the horizontal. The full image arrangement for the case $\alpha = \pi/4$ is shown in figure 8. One can take (Rutherford 1959, p. 51) the complex potential of a dipole of unit strength at $z = z_0$ as $w_0 = \mathrm{e}^{\mathrm{i}\beta}/(z - z_0)$, in which case the potential of the full image system shown in the diagram (satisfying Neumann conditions on both $\theta = 0$ and $\theta = -\pi/4$) is

$$w_0(z) = \frac{4z_0^3 \mathrm{e}^{\mathrm{i}\beta}}{z^4 - z_0^4} + \frac{4\overline{z_0}^3 \mathrm{e}^{-\mathrm{i}\beta}}{z^4 - \overline{z_0}^4}.$$

Taking the opportunity to generalize also to beach slope $\alpha$, the real-valued velocity

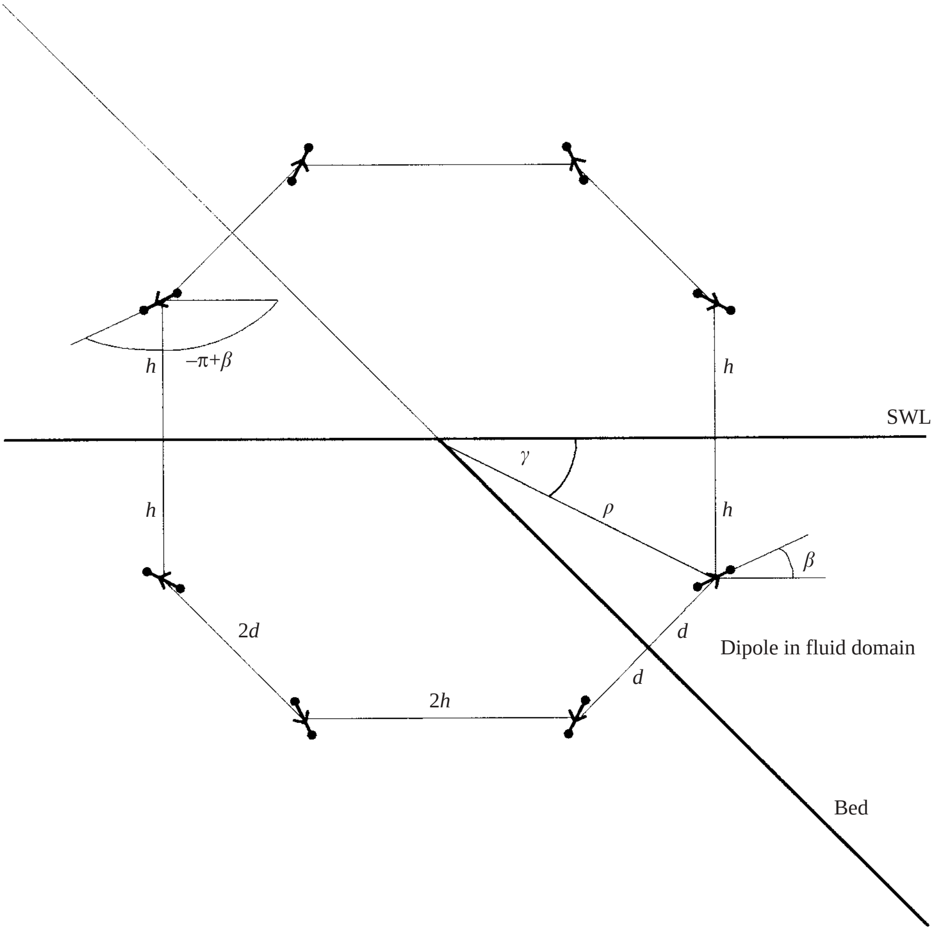


FIGURE 8. Submerged dipole: image distribution in the case $\alpha = \pi/4$, $d = \rho \sin(\alpha - \gamma)$.

potential may be written in the form

$$\phi(R, \theta) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} R^{-s} D(s) \Gamma(s) \cos s(\theta + \alpha)\, \mathrm{d}s + \mathrm{Re}\{w_0(z)\}. \tag{8.1}$$

where $z_0 = \rho \mathrm{e}^{-\mathrm{i}\gamma}$ and where now

$$w_0(z) = \frac{\pi}{\alpha} \left\{ \frac{z_0^{\pi/\alpha - 1} \mathrm{e}^{\mathrm{i}\beta}}{z^{\pi/\alpha} - z_0^{\pi/\alpha}} + \frac{\overline{z_0}^{\pi/\alpha - 1} \mathrm{e}^{-\mathrm{i}\beta}}{z^{\pi/\alpha} - \overline{z_0}^{\pi/\alpha}} \right\}.$$

The correct surface condition is to be satisfied by appropriate choice of $D(s)$ although now, of course, there is no need to assign a double pole of the integrand at $s = 0$. By writing

$$D(s) = \frac{B(s) d(s) \sin \pi s}{\cos s\alpha}$$

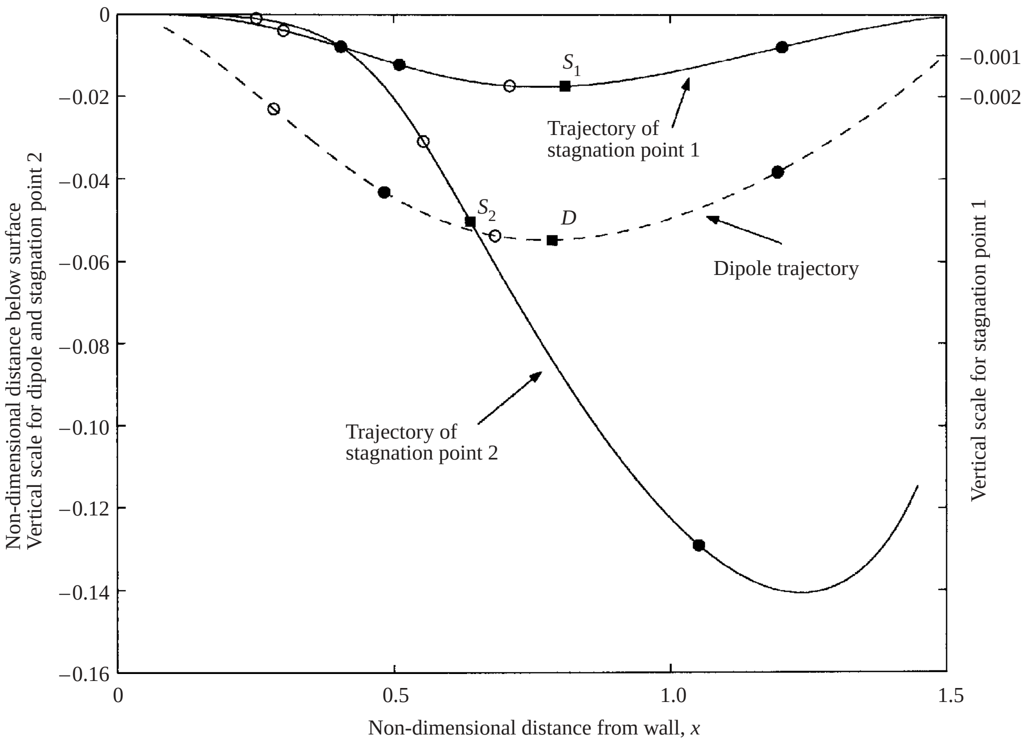


FIGURE 9. Submerged dipole: dipole and associated stagnation-point trajectories. Note different scales for $S_1$, $S_2$. Case: $\alpha = \pi/2$; vertical wall at $x = 0$. McIver's is case shown with squares $D$, $S_1$, $S_2$.

and after noting the Mellin transform of $\mathrm{Re}\{(z_0^{\pi/\alpha-1}\mathrm{e}^{\mathrm{i}\beta})/(z^{\pi/\alpha} - z_0^{\pi/\alpha})\}$ the equation satisfied by $d(s)$ is found to be

$$d(s) - d(s-1) = \frac{2\pi\rho^{s-1}\cos(\beta+\gamma-s(\gamma-\alpha))}{\Gamma(s)B(s)\sin s\pi\sin(s\alpha)}.$$

Since it is now required for $d(s)$ to be regular at the origin, the solution, following previous procedures, can be expressed in the form

$$d(s) = 2\pi\int_{3/4-\mathrm{i}\infty}^{3/4+\mathrm{i}\infty}\frac{\cos(\beta+\gamma-\tau(\gamma-\alpha))\rho^{\tau-1}}{\sin\tau\alpha\sin\pi\tau\,\Gamma(\tau)B(\tau)}\left[\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}(s-\tau)}}\right]\mathrm{d}\tau. \qquad (8.2)$$

The requirement is that $|d(s)| = \mathrm{O}(|s|^{-1})$ as $\mathrm{Im}\,s \to \pm\infty$ in order that there are no radiating waves. The behaviour of $d$ is clearly exponentially small as $\mathrm{Im}\,s \to -\infty$ so it remains only to ensure that the integral expression in equation (8.2) has the required behaviour as $\mathrm{Im}\,s \to +\infty$. Noting the convolution $f(s)f(1-s) = \pi$ cosec $\pi s$ satisfied by both $B(s)$ and $\Gamma(s)$ it follows that

$$f_k(\rho) = \int_{1/4-\mathrm{i}\infty}^{1/4+\mathrm{i}\infty}\rho^{-1-\tau}B(\tau)\Gamma(1+\tau)\sin\pi\tau\cos(\beta+\gamma+\tau(\gamma-\alpha))\sec\tau\alpha\,\mathrm{d}\tau = 0. \quad (8.3)$$

This could also be verified by placing equal but opposite sources close to each other

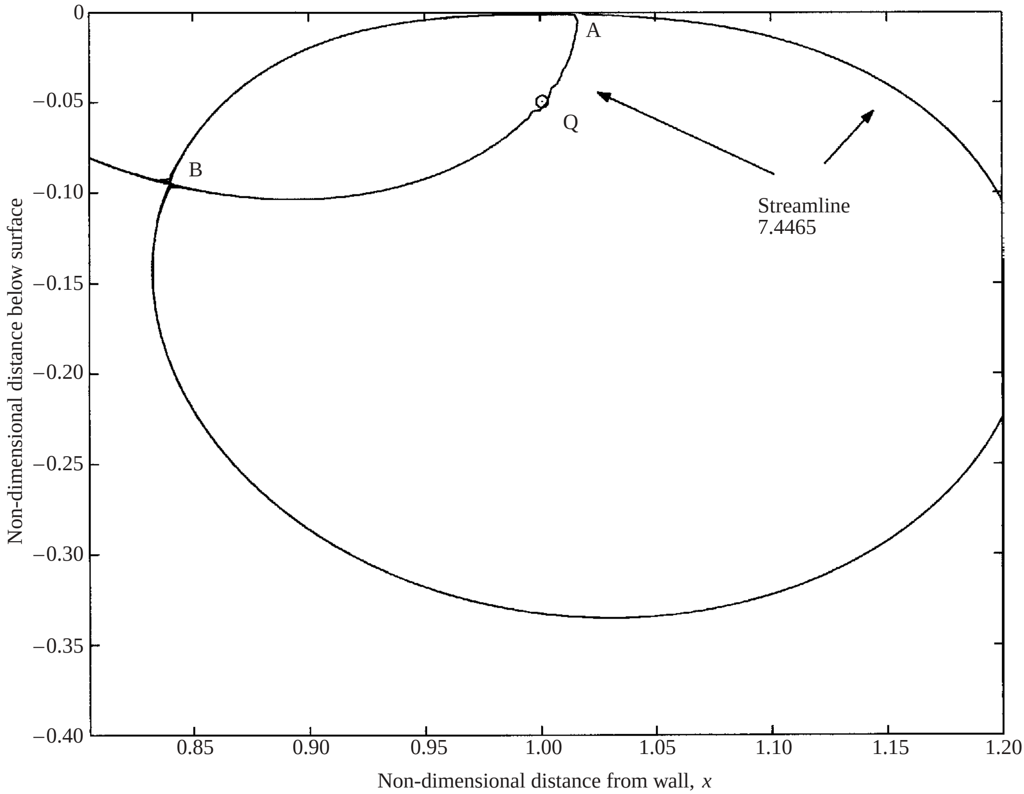


FIGURE 10. Streamline around submerged dipole. Case: $\alpha = \pi/2$; vertical cliff; dipole position Q (1.0013, −0.0496), upper stagnation Point A (1.018, −0.00136), lower stagnation Point B (0.8395, −0.0950).

anywhere on any one of the curves of zero potential for the classical standing wave problem. A dipole whose axis is tangential to the chosen curve is thereby mimicked. Morris (1974*a*) sketched some of these curves for various beach angles which in polars would be given parametrically by setting $\phi_0(\rho, -\gamma) = 0$ in equation (6.3). The dipole axis would then be orthogonal to $\nabla\phi_0$. Thus, forming $\mathbf{ds}\cdot\nabla\phi_0 = 0$, where $\mathbf{ds}$ is along the dipole axis and therefore parallel (in polars) to the vector $(\cos(\beta+\gamma), \sin(\beta+\gamma))$, one recovers equation (8.3) from this condition.

In order to relate the above result to the recent work of McIver (2000), a special case is now considered.

### 8.1. *Special case of a vertical cliff*

It is interesting to compare the works of Morris (1974*a*) where, for the vertical cliff case, it was noted that, to achieve trapped waves, the single surface source could be placed at zeros of $\cos x$, while McIver (2000) placed submerged dipoles on the vertical line $x = \pi/4$ albeit very close to the surface. With her choice $\kappa a = \pi/4$, McIver is essentially combining two equal-strength horizontal and vertical dipoles in front of a vertical cliff, in such a way that this would reduce to a single dipole with axis given by $\beta = \pi/4$ in the present model. It follows (for $k = 1$) that $B(s) = (\sqrt{2\pi}\cos s\alpha)/(\sin \pi s)$ and hence from equation (8.3) that the requirement of non-radiating waves on the

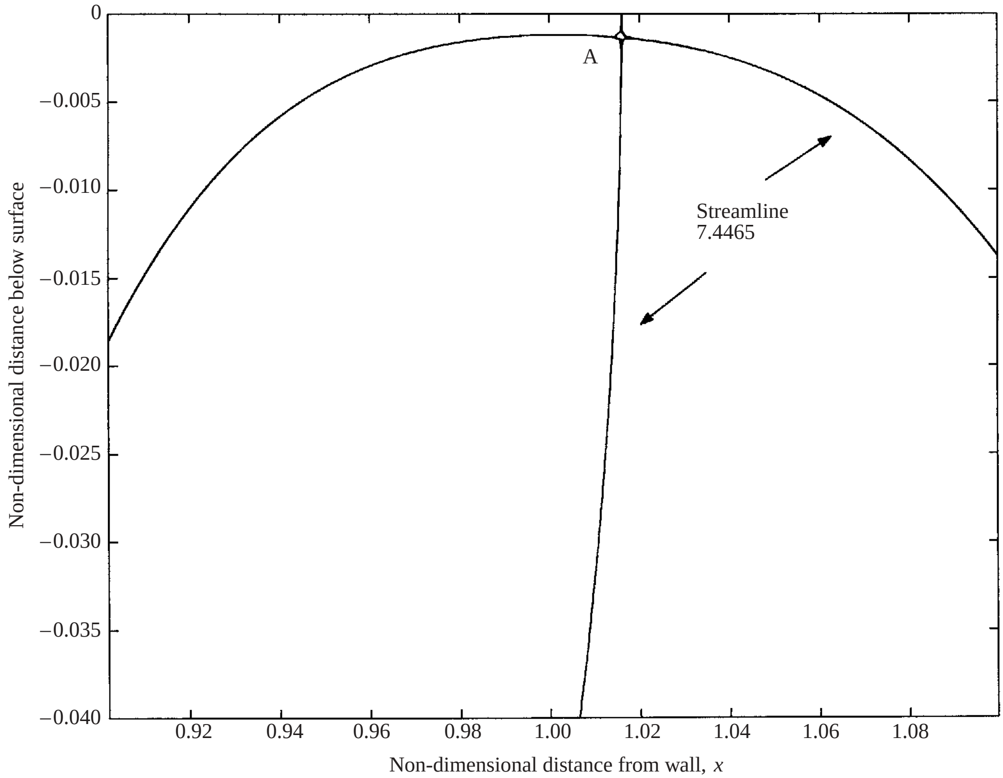


FIGURE 11. Streamline around submerged dipole: a near-surface view of the case shown in figure 10.

chosen line can be written

$$\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \rho^{-s}\Gamma(s)\cos(\beta+\pi/2+s(\gamma-\alpha))\,\mathrm{d}s = \mathrm{e}^{-\rho\cos(\gamma-\alpha)}\cos(\rho\sin(\gamma-\alpha)+\pi/2+\beta) = 0,$$

following the transformation $s = \tau + 1$. For the vertical cliff ($\alpha = \pi/2$), this gives simply $\rho\cos\gamma - \beta = 0$, so on the line chosen by McIver, the dipole axis required is consistent with $\beta = \pi/4$. Moreover, other lines $x =$ const. can be chosen, provided the dipole axis is rotated accordingly. Of course, McIver chose $x = \rho\cos\gamma = \pi/4$ because there she was able to observe a closed streamline surrounding the source. The present investigation will reveal numerically that, provided the dipole is rotated axially, such closed streamlines do actually exist for a whole range of values of dipole locations. Evidence of this is demonstrated below. Meanwhile, it is confirmed (as a benchmark test) that the numerical results of McIver are indeed reproduced by the present model.

8.1.1. *Stagnation points*

The location of stagnation points is paramount in a quest for a closed submerged streamline. McIver (2000) showed that the coordinates of the two points that exist, in her example, are respectively $S_1$ : (0.809116, −0.00174415) and $S_2$ : (0.637412,

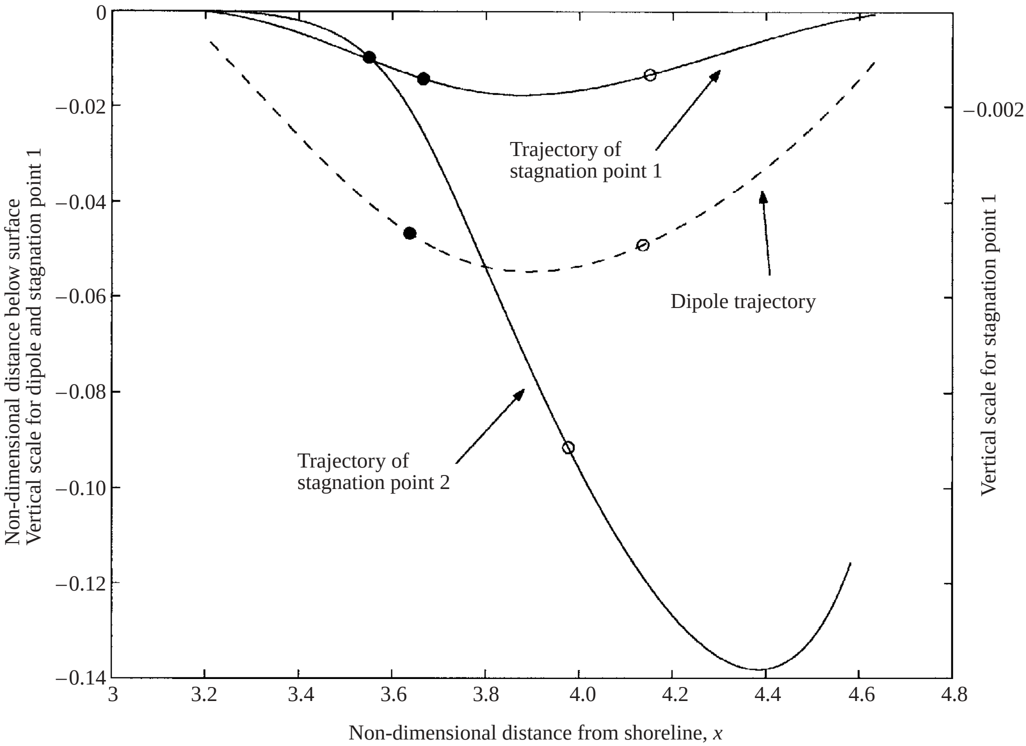


FIGURE 12. Submerged dipole: dipole and associated stagnation-point trajectories. Case: $\alpha = \pi/2$; vertical wall at $x = 0$, second (seaward) mode. Symbols show two random dipole positions with corresponding stagnation points.

$-0.0499727$) and that the common stream function value at these points is $\psi = 6.97737$. In order to seek a numerical procedure to determine these, the full complex potential function $w(z)$ (retaining the form for general angles $\alpha$) is written

$$w(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s} D(s)\Gamma(s)e^{-is(\theta+\alpha)}\,ds + w_0(z). \tag{8.4}$$

Differentiation shows that

$$zw_0'(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s} D(s)\Gamma(s+1)e^{-is(\theta+\alpha)}\,ds \tag{8.5}$$

at a stagnation point.

By accepting the convenience of integer $k$ the closed-form solution for $B(s)$ may be used (to simplify computation) and with the conjugate of equation (8.1) the stream function may be written

$$\psi(R,\theta) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s} D(s)\Gamma(s)\sin s(\theta+\alpha)\,ds + \frac{\pi}{\alpha}\mathrm{Im}\{w_0(z)\}. \tag{8.6}$$

An expression for $d$ which can be used on the line of integration $\mathrm{Re}(s) = \frac{1}{2}$ may be written either from residue calculation or using Titchmarsh's method to solve the

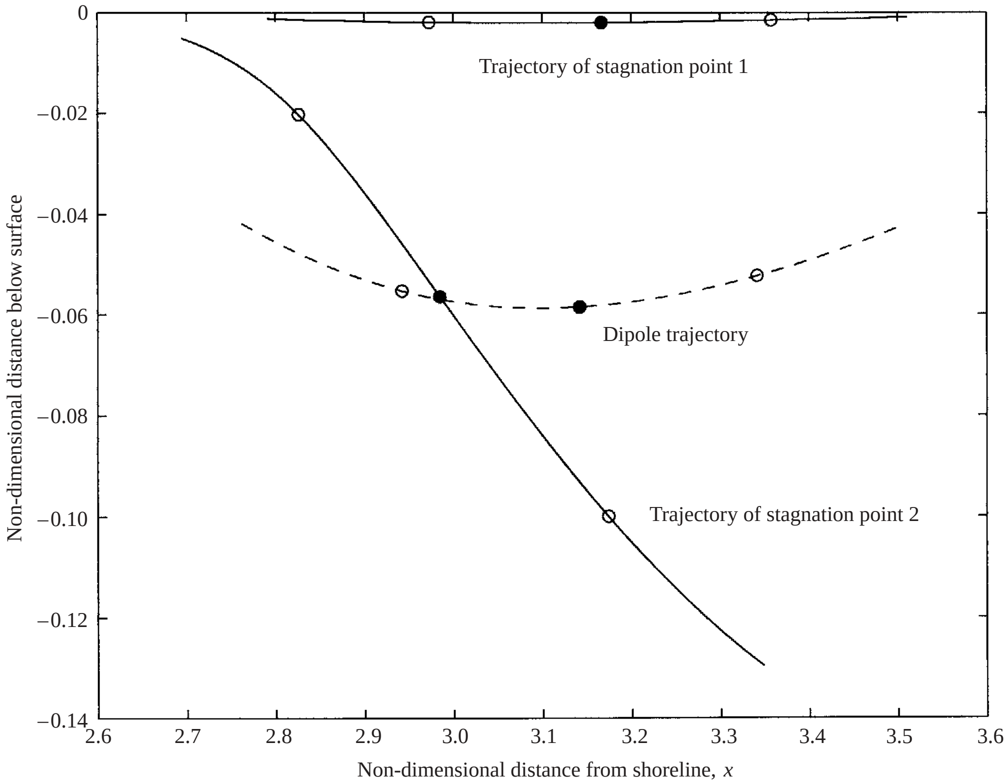


FIGURE 13. Submerged dipole: dipole and associated stagnation-point trajectories. Case: $\alpha = \pi/4$; beach gradient unity. Note that vertical scale is the same for each trajectory. Symbols alternate from left to right to show corresponding points.

difference equation. This gives

$$d(s) = 2\pi\rho^{s-1}\sum_{N=1}^{\infty}\frac{\rho^N\cos(\beta+\gamma-(s+N)(\gamma-\alpha))}{\Gamma(s+N)B(s+N)\sin(s+N)\pi\sin(s+N)\alpha}, \quad 0 < \mathrm{Re}(s) < 1. \tag{8.7}$$

By keeping to complex potential form, a complex Newton–Raphson iterator is used to determine stagnation points and thereby track, for different dipole positions, the stream function values at these points; in particular mapping the value of $\delta\psi = \psi_1 - \psi_2$ where $\psi_i$ is the value at the $i$th stagnation point. McIver found the closed streamline whenever $\delta\psi = 0$. The sensitivity of the numerical computations is extreme, partly due to the proximity of the free surface (which cannot be traversed in the Newton–Raphson iterations); however, with McIver's values as starting values an almost complete trajectory of dipole positions on $0 < x < \pi/2$ has been found. This is displayed in figure 9 along with the trajectories of associated stagnation points $S_1, S_2$ (note the different scales used for these two points). Alternating symbols are inserted at four random positions (square indicates McIver's parameters) to show the association between a local dipole position and the points $S_1, S_2$.

To investigate the nature of the closed streamline surrounding the dipole, the horizontal coordinate 1.0013 is chosen (quite randomly) for the dipole (Q) yielding a computed vertical coordinate (−0.0496, which gives no radiated waves), the associated

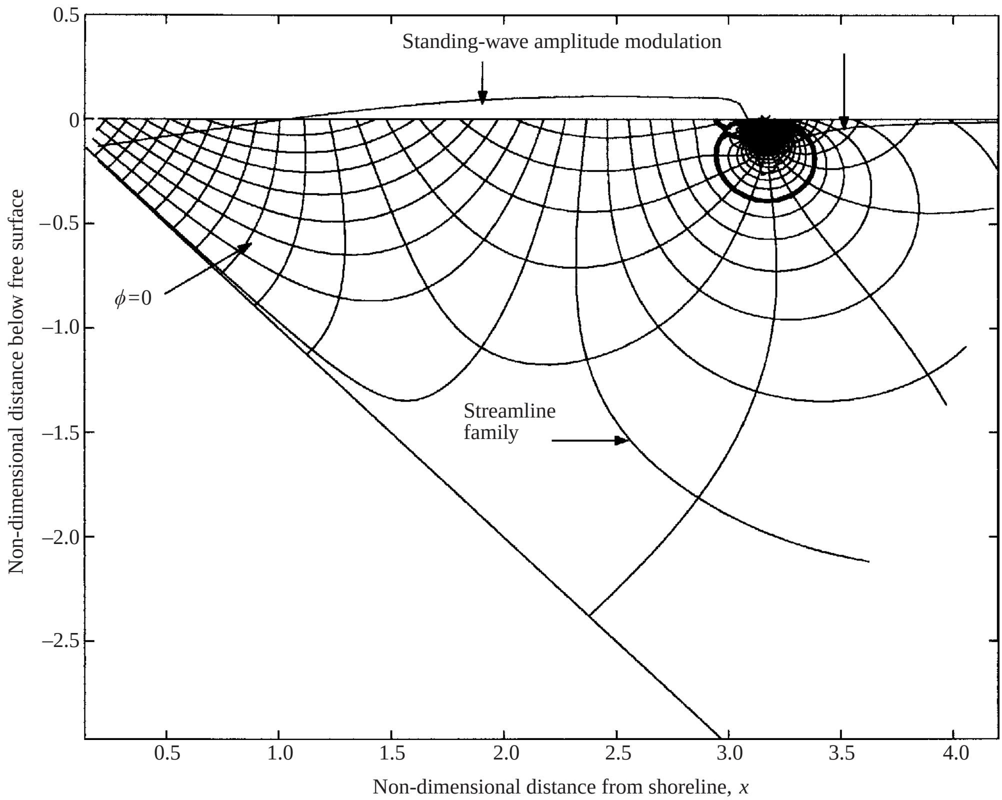


FIGURE 14. Streamlines and equipotentials around submerged dipole. Case: $\alpha = \pi/4$. Equipotentials and streamlines for oscillating dipole placed at ($\pi$, $-0.058$). Thick line denotes closed submerged streamline $\psi = -7.146452$.

stagnation points A and B, the common stream function value at these points ($\psi = 7.4465$); plotted in figures 10 and 11 is the closed streamline. Figure 11 is a blow-up showing the near-surface view so that the submerged nature of the streamline is more clearly visible.

The occurrence, in the case of surface sources as discussed earlier, of non-radiating source positions at a denumerably infinite set of points prompts a further investigation for the case of submerged dipole. It was seen that, for this vertical wall case, the dipole trajectory appears to enter the free surface from below (and therefore essentially disappear) near both $x = 0$ and $x = \pi/2$. McIver (2000) used the notion of virtual streamlines above the free surface and one can similarly imagine the dipole trajectory becoming virtual at $x = \pi/2$ and, if oscillatory, set to return below the free surface (for a mode 2 contribution, say) possibly at $x = \pi$. This has been confirmed to a certain extent numerically and there seems no reason why this behaviour should not continue further seaward also. The numerical evidence is presented in figure 12 (details similar to the previous figures).

## 9. Computations for a beach

Both the cases of a source on the free surface and a submerged dipole have been computed in the case of a beach of gradient unity. The difficulty in computing

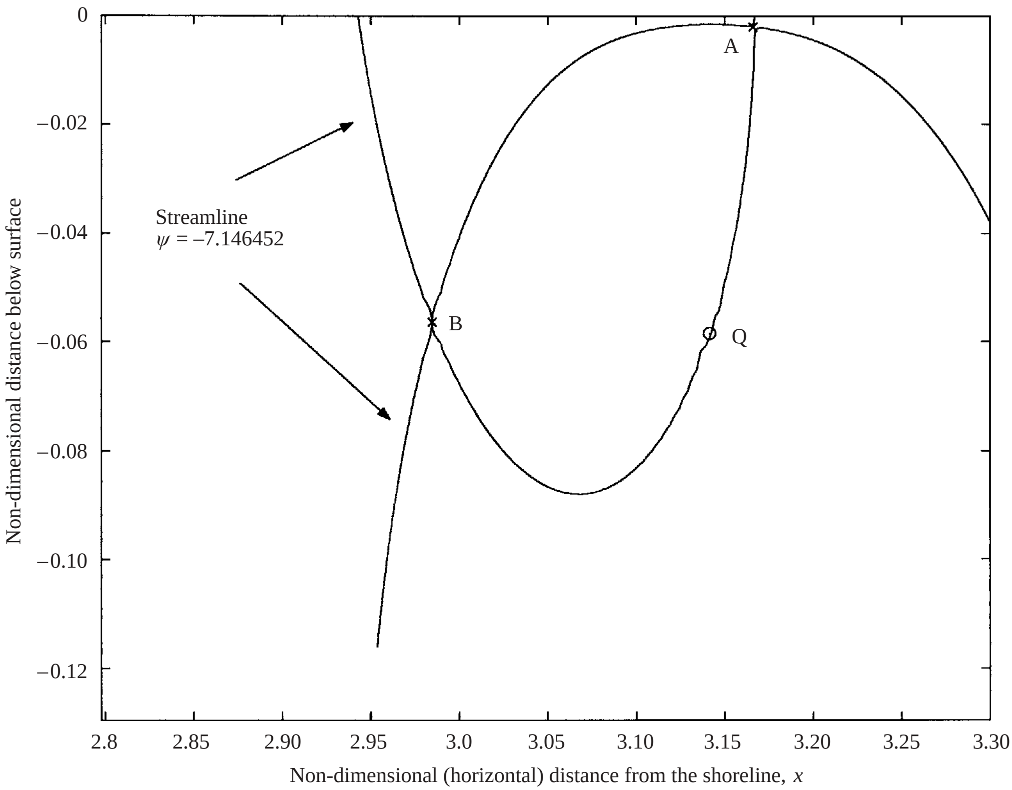


FIGURE 15. Streamline around submerged dipole: a near-surface view of the case shown in figure 14. Upper stagnation point A (3.166872, −.002005). Lower stagnation point B (2.984664, −.056376).

similarly for shallower beaches of slope $r\pi/2n, r, n \in \mathbf{Z}, (r, n) = 1$ would only be the extra algebra required to solve equation (8.3) for the positions of the dipole, whilst for more general angles, the summation used within the inversion integral (e.g. (7.5)) would need to be replaced by a further infinite integral. Such routines have been successfully carried out by the author in related work, e.g. Ehrenmark (1987).

### 9.1. *Dipole within the fluid domain*

Calculations similar to those described in the previous section (for the vertical wall) have been applied to the case of a beach of gradient unity. Having established in the wall case that there was a range of values of the $x$-coordinate for which the pivotal two stagnation points could be identified, it was more straightforward to locate these in the beach case. However, for the beach of unity gradient, it is found that the primary domain of existence seems to have shifted from $x \in [0, \pi/2]$ to $x \in [3\pi/4, 5\pi/4]$ (see figure 13).

Having computed full data for the stagnation points and dipole positions possible, it is then an easy matter to identify a closed submerged streamline surrounding any one of these chosen dipoles. Shown in figures 14 and 15 is the case where the dipole is placed on the vertical line $x = \pi$. The (unique) vertical position which guarantees zero radiation is found to be $y = -0.058$. The required orientation of the dipole turns out to be given by $\beta = -2.35339$ radians. Determining the orientation is not entirely trivial. One can note the exact solution for $f_k(\rho)$ from equation (8.3) by using the appropriate form of $B(s)$ using equation (6.5) in the case of integer $k(= n)$. For the case of the beach of unit gradient ($k = 2$), it is found that

$$\tan(\beta + \alpha) = \frac{\mathrm{e}^x \cos x + \mathrm{e}^y \cos y}{\mathrm{e}^y \sin y - \mathrm{e}^x \sin x}, \tag{9.1}$$

where the dipole position is $z = x - \mathrm{i}y$. Figure 14 displays both streamlines and equipotentials together with the outline of the submerged closed streamline (shown with a bold line). The submerged nature of this contour is more clearly shown in the 'blow-up' in figure 15. Also computed is the equivalent standing-wave modulation profile (for a relative amplitude 0.01). This shows clearly the occurrence of a node between the shoreline and the equivalent obstacle and also the decay of the wave amplitude seaward of the obstacle. One would anticipate further nodes entering if the dipole position is moved seaward to the next range of existence of the stagnation point pair, and so on.

## 10. Conclusion

It is demonstrated that the pioneering work on wave trapping due to surface-piercing or submerged obstacles by McIver (1996, 2000) can be extended from uniform or infinite depth to the case where the bed is a plane incline. It is found here that the situations created by surface sources and submerged dipoles are essentially different. For the former, infinitely many closed streamlines exist for each of a denumerably infinite set of source positions which happen to be the zeros of potential in the classical perfect reflection problem. Each of the streamlines enclose the source so that any one can be chosen to mimic the presence of a solid body. For the submerged dipole it is found instead that only one closed streamline exists, but that this can be constructed for each of a non-denumerably infinite set of dipole positions. Whilst the theory is applicable to beaches of any slope, for expedience in the present work the computations have been limited to beach angles of the form $\pi/2k$ for integer $k$. It is likely that the case of an overhanging cliff for which $k < 1$ would also be of interest and further development of numerical quadrature routines should reveal this. Equally, additional experimentation with pairs of submerged unit sources of opposite sign could be interesting. If suitably placed on distinct 'non-radiative' trajectories (as discussed earlier) there seems to be no immediate physical reason why such a pair (now not forming a dipole) should not also, in some cases, be surrounded by a closed streamline. The present author has hitherto been unable to support this conjecture with evidence and further experimentation is desirable.

Work is being undertaken to explore the possibility of extending this theory to the case of oblique waves. Potentials for this three-dimensional problem have been constructed by Morris (1976) using a method of Green's functions and it remains to be seen if these can be used to develop a similar theory for oblique wave trapping by surface-piercing or submerged cylinders. Meanwhile, readers wishing to experiment further with the two-dimensional cases may be interested in a 'rough and ready' Fortran77 program available for at least 12 months after the publication date of the present article at www.lgu.ac.uk/cismres/xtra/jfm03.txt. One question that might be explored is this: Given that, in each modal range, there exists a unique maximally submerged closed stream-line (and therefore, by implication, a uniquely shaped maximally submerged obstacle) how do the sizes and locations of these obstacles vary as the mode number increases (i.e. as the implied obstacle is taken further and further seawards)?

Thanks are due to Maureen McIver and Richard Porter for a number of extremely helpful discussions and also verification of numerical results on the vertical wall case (M.M.), to D. V. Evans for suggesting and discussing the problem and also to the anonymous referees for their many useful comments.

## Appendix

Some of the results in the text require a careful analysis of the zeros and poles of the function $B(s)$ defined by equation (6.5) in the text. In Ehrenmark (1987) it is shown that

$$B(s) = \Gamma(s) \exp F_k(s), \quad -k < \mathrm{Re}(s) < k+1$$

where $F_k(s)$ is analytic in this strip. Thus $B(s)$ has a simple pole at $s = 0$ but is otherwise regular and non-zero in $0 < \mathrm{Re}(s) < k+1$. From the recurrence relation $B(s+1) = B(s) \tan s\alpha$ it therefore follows that $B$ has a pole at $s = k+1$. If the recurrence is applied continually, it can be seen that poles appear at all points $s = k+j$, $j = 1, 2\ldots$ unless $k$ is an integer, in which case zeros appear at $s = 2k+j$, $j = 1, 2\ldots$, to cancel the poles. There are further sets of poles of this type at $s = 3k+j$, $j = 1, 2\ldots$, $s = 5k+j$, $j = 1, 2\ldots$ and so on (which again are terminated by zeros at $s = 4k+j$, $j = 1, 2\ldots$, $s = 6k+j$, $j = 1, 2\ldots$, when $k$ is integer).

It may be noted, when $k$ is integer, that $1/(B(s) \sin s\alpha)$ is analytic in the entire right-hand half-plane. This observation facilitates the construction of $d(s)$ as an infinite summation, rather than an infinite integral. In an earlier work, Ehrenmark (1988), it is shown that, even in the case of irrational $k$, a meaningful summation can be constructed provided this is interpreted in the sense of over-convergence (see also Titchmarsh 1939, p. 220). On the other hand, it is also perfectly possible to compute $d(s)$ interactively by numerical quadrature. For the submerged poles and dipoles, the defining integral converges at an exponential rate as $|s| \to \infty$ on $\mathrm{Re}s = \frac{1}{2}$ with the exponent proportional to the polar angle of the submergence.

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